Week 05 | Lecture 05
Mathematical Foundations

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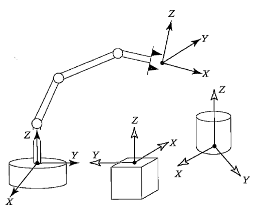
Robot of the Day

https://www.youtube.com/watch?v=tIIJME8-au8&feature=emb_title

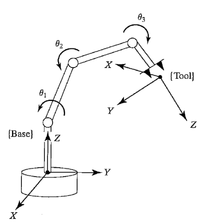
Mechanics & Control

A Typical Knowledge Flow of Mechanical Manipulators

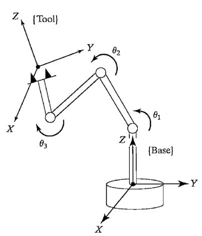
Description of Position and Orientation



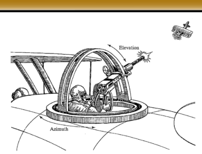
Forward Kinematics of Manipulators



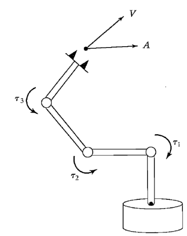
Inverse Kinematics of Manipulators



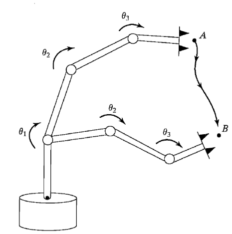
Velocities, Static Forces, Singularities



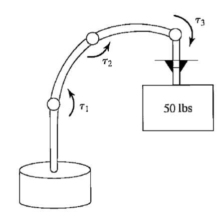
Dynamics



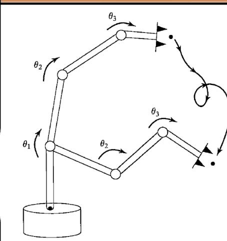
Trajectory Generation



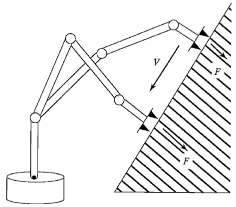
Manipulator Design and Sensors



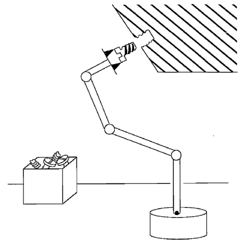
Linear & Nonlinear Position Control



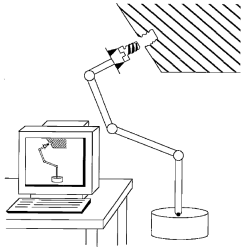
Force Control

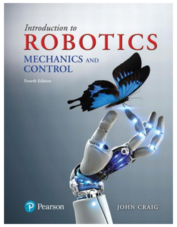


Programming Robots



Off-line Programming and Simulation





Introduction to Robotics: Mechanics and Control

By John Craig

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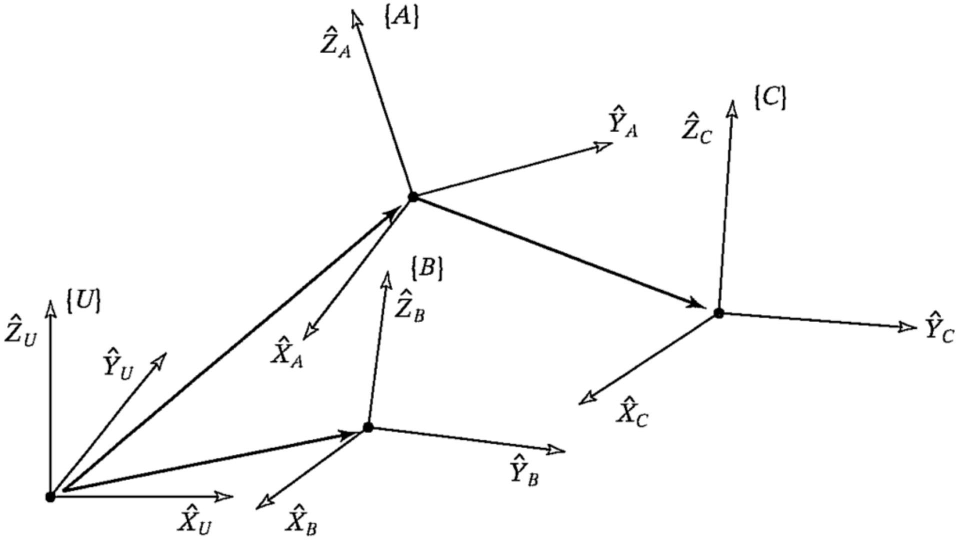
A Universe Coordinate System

The assumption that everything is referenced to …

- A coordinate system attached to the World Frame

- All positions and orientations w. r. t.
  - The Universe Coordinate System, or
  - Other Cartesian CS that are (or could be) defined relative to the Universe CS.

- **Robotic Mechanisms** are systems of rigid bodies connected by joints.
- **Pose** is the collective term of the *position* and *orientation* of a rigid body in space.



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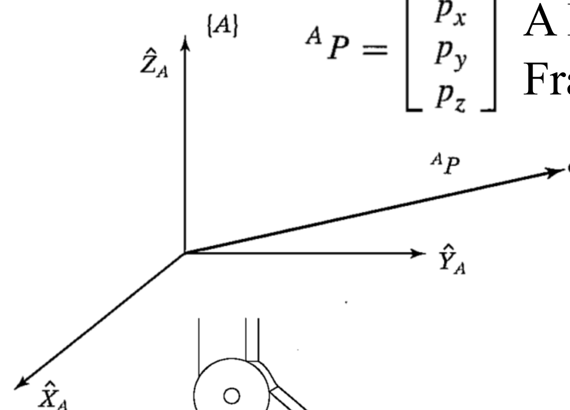
Position & Orientation

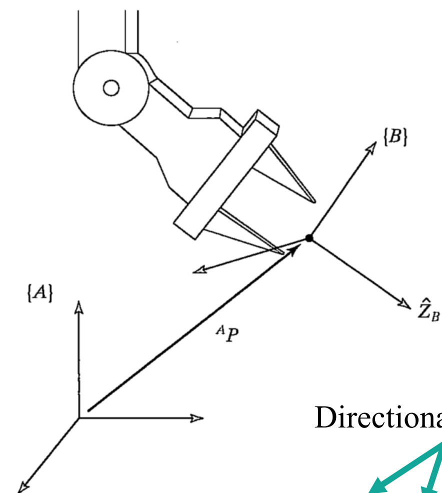
For $p \in \mathbb{R}^n$, $n = 2$ for planar, $n = 3$ for spatial

- Point: $p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$, $\|p\| = \sqrt{p_1^2 + \cdots + p_n^2}$

$${}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$ A Point in Frame {A}



- Vector: $v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ \vdots \\ p_n - q_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$



Rotation of Frame {B} relative to Frame {A}

- Matrix: $A \in \mathbb{R}^{n \times m}$, $A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$

Directional cosines

$${}^{A}_{B}R = \left[\, {}^{A}\hat{X}_B \;\; {}^{A}\hat{Y}_B \;\; {}^{A}\hat{Z}_B \,\right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$${}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T}$$

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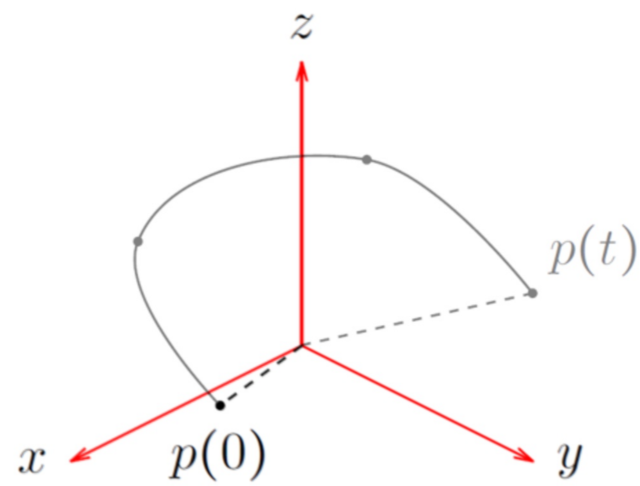
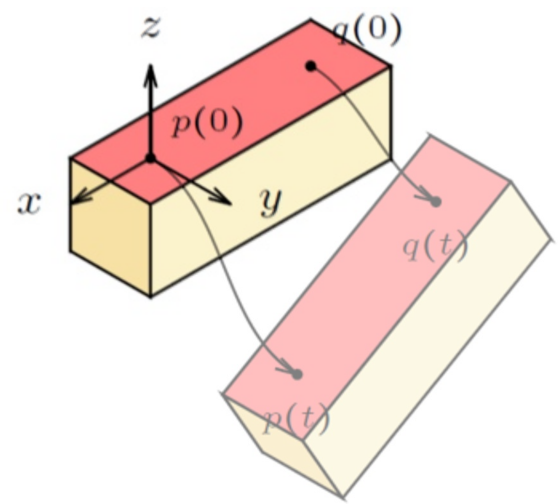
Description of Point-Mass Motion

Rigid-body Assumption

- $p(0) = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}$: initial position
- $p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, t \in (-\epsilon, \epsilon)$



- Trajectory
  - A curve $p: (-\epsilon, \epsilon) \longmapsto \mathbb{R}^3, p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$
- Rigid body transformation
  - $\|p(t) - q(t)\| = \|p(0) - q(0)\| = constant$



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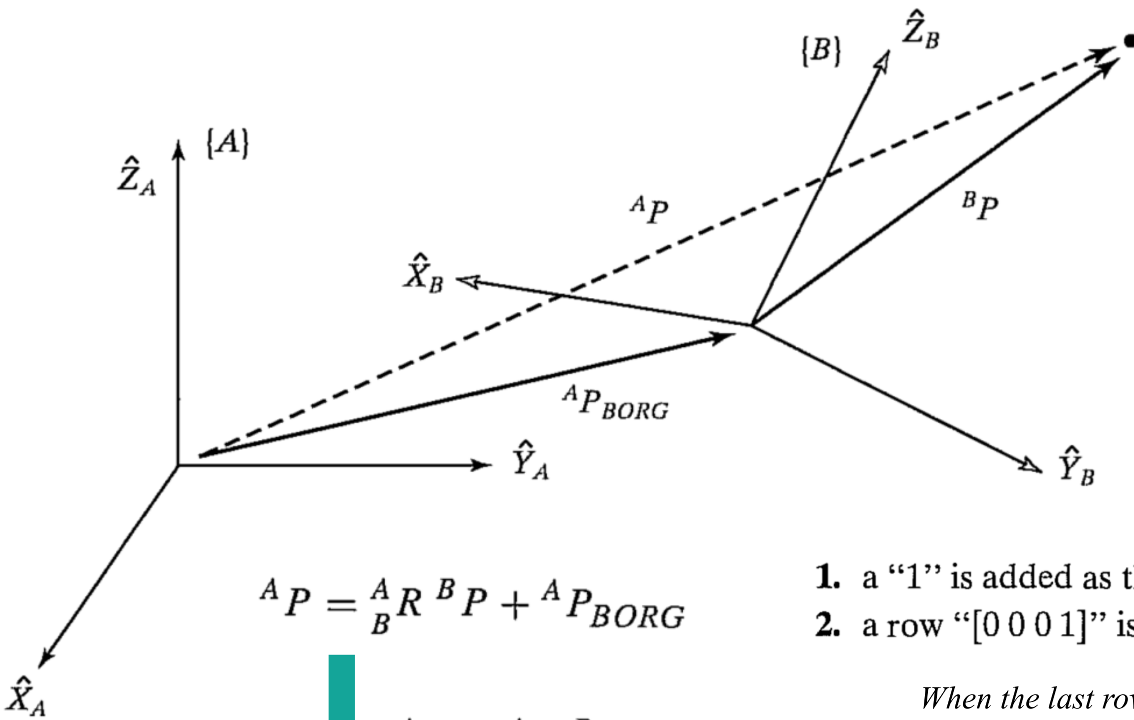
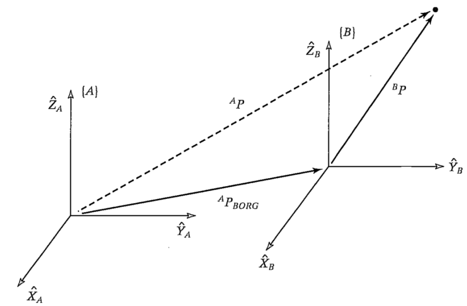
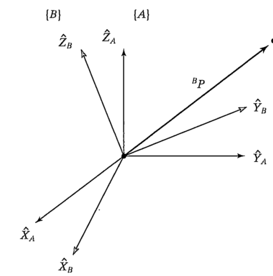
Homogeneous Transformation

Translation + Rotation = Transformation







$${}^AP = {}^A_BR\,{}^BP + {}^AP_{BORG}$$

$${}^AP = {}^A_BT\,{}^BP$$

1. a "1" is added as the last element of the 4 × 1 vectors;
2. a row "[0 0 0 1]" is added as the last row of the 4 × 4 matrix.

When the last row is other than [0 0 0 1] or the rotation matrix is not orthonormal, this 4x4 matrix can be also used to compute perspective and scaling operations

$$\begin{bmatrix} {}^AP \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^A_BR & & {}^AP_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^BP \\ 1 \end{bmatrix}$$

$${}^AP = {}^A_BR\,{}^BP + {}^AP_{BORG}$$
$$1 = 1.$$

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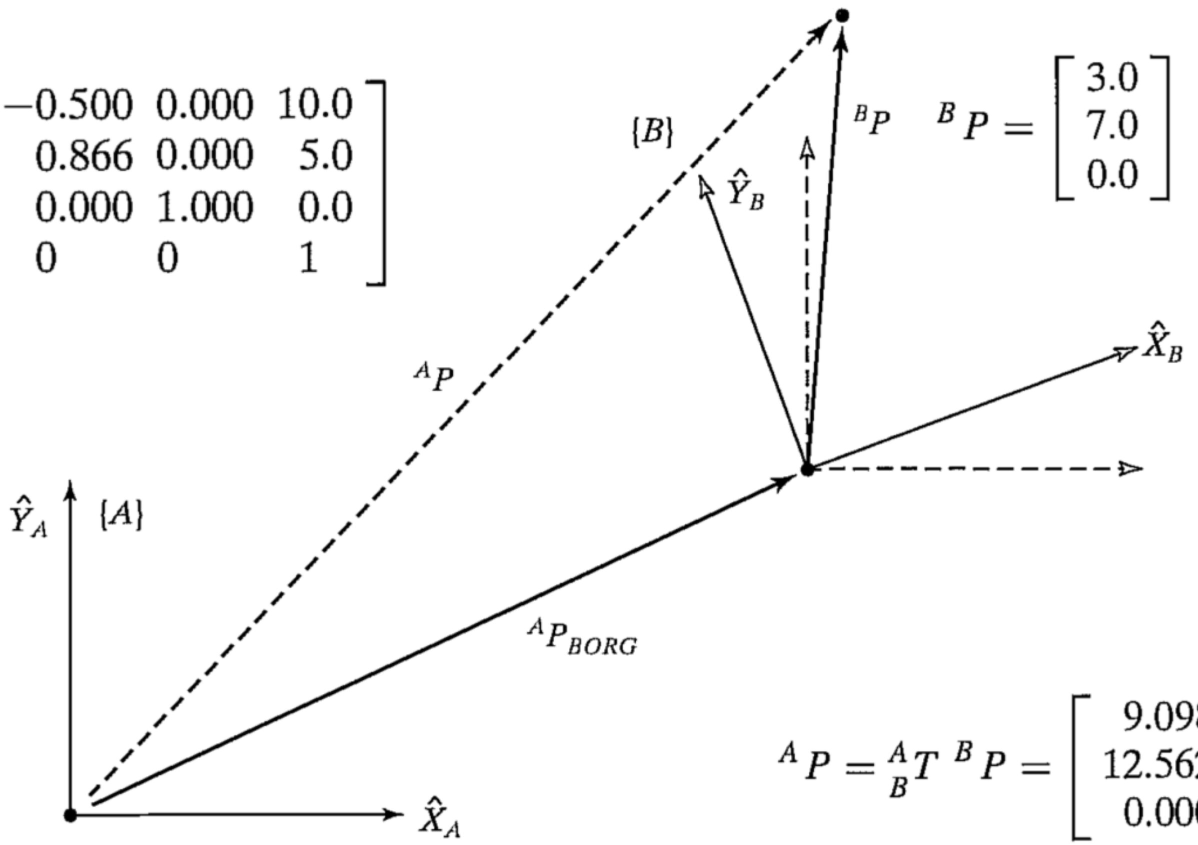
Exercise

Figure 2.8 shows a frame $\{B\}$, which is rotated relative to frame $\{A\}$ about $\hat{Z}$ by 30 degrees, translated 10 units in $\hat{X}_A$, and translated 5 units in $\hat{Y}_A$. Find ${}^{A}P$, where ${}^{B}P = [3.0\,7.0\,0.0]^T$.

$$
{}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



$$
{}^{B}P = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix}
$$

$$
{}^{A}P = {}^{A}_{B}T\,{}^{B}P = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix}
$$

Interpretations of Transformation

Three ways

- **It is a description of a frame.**
  - ${}^{A}_{B}T$ describes the frame {B} relative to the frame {A}. Specifically, the columns of ${}^{A}_{B}R$ are unit vectors defining the directions of the principal axes of {B}, and ${}^{A}P_{BORG}$ locates the position of the origin of {B}.

- **It is a transform mapping.**
  - ${}^{A}_{B}T$ maps ${}^{B}P \rightarrow {}^{A}P$

- **It is a transform operator.**
  - T operates on ${}^{A}P_1$ to create ${}^{A}P_2$

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Common Operators

Translational / Rotational / Transformation

- Translational Operator

$${}^{A}P_2 = D_Q(q)\,{}^{A}P_1$$

$$\mathrm{Trans}(a,b,c) = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Rotational Operator

$${}^{A}P_2 = R_K(\theta)\,{}^{A}P_1$$

$$\mathrm{Rot}_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{Rot}_y(\theta_y) = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{Rot}_z(\theta_z) = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Transformation Operator

$${}^{A}P_2 = T\,{}^{A}P_1$$

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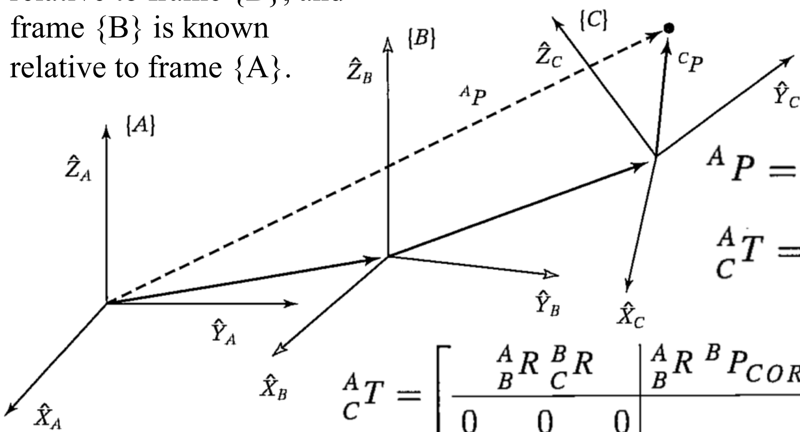
Transformation Arithmetic

Compound / Inversion / Equation

Frame {C} is known relative to frame {B}, and frame {B} is known relative to frame {A}.



$$^{A}P = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P$$

$$^{A}_{C}T = {}^{A}_{B}T\,{}^{B}_{C}T$$

$$^{A}_{C}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R\,{}^{B}_{C}R & & {}^{A}_{B}R\,{}^{B}P_{CORG} + {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

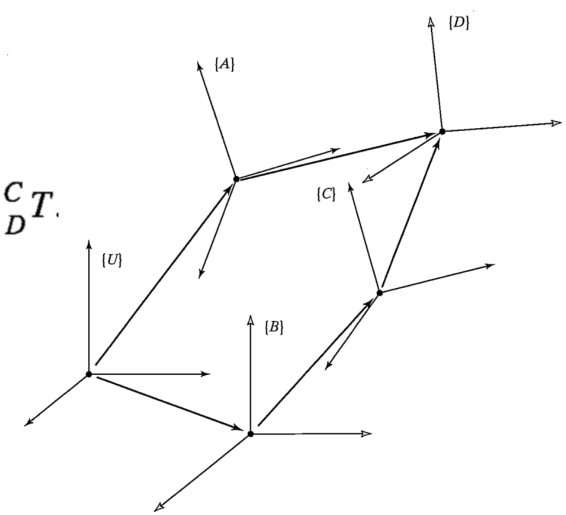
Avoid direct inverse operation

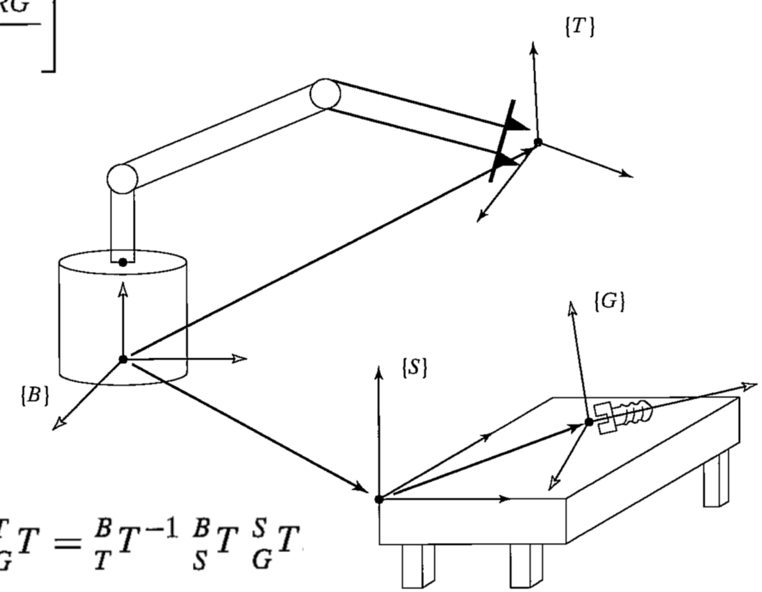
- Computationally expensive in practice

A general and extremely useful way of computing the inverse of a homogeneous transform.

$$^{B}_{A}T = {}^{A}_{B}T^{-1} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$^{U}_{A}T\,{}^{A}_{D}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T.$$





$$^{T}_{G}T = {}^{B}_{T}T^{-1}\,{}^{B}_{S}T\,{}^{S}_{G}T$$

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Proper Orthonormal Matrix

Det = +1 Orthogonal + Normalized

- Cayley's Formula for orthonormal matrices

a skew-symmetric matrix

Any proper orthonormal matrix

$$R = (I_3 - S)^{-1}(I_3 + S)$$

$$S = \begin{bmatrix} 0 & -s_x & s_y \\ s_x & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}$$

  - *Any 3 x 3 rotation matrix can be specified by just 3 parameters*

- But rotations don't usually commute

$$R_z(30) = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \qquad R_z(30)R_x(30) = \begin{bmatrix} 0.87 & -0.43 & 0.25 \\ 0.50 & 0.75 & -0.43 \\ 0.00 & 0.50 & 0.87 \end{bmatrix}$$

$$R_x(30) = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.866 & -0.500 \\ 0.000 & 0.500 & 0.866 \end{bmatrix} \qquad \neq R_x(30)R_z(30) = \begin{bmatrix} 0.87 & -0.50 & 0.00 \\ 0.43 & 0.75 & -0.50 \\ 0.25 & 0.43 & 0.87 \end{bmatrix}$$

- ***<u>How to construct a simpler representation with the minimal (three) numbers?</u>***

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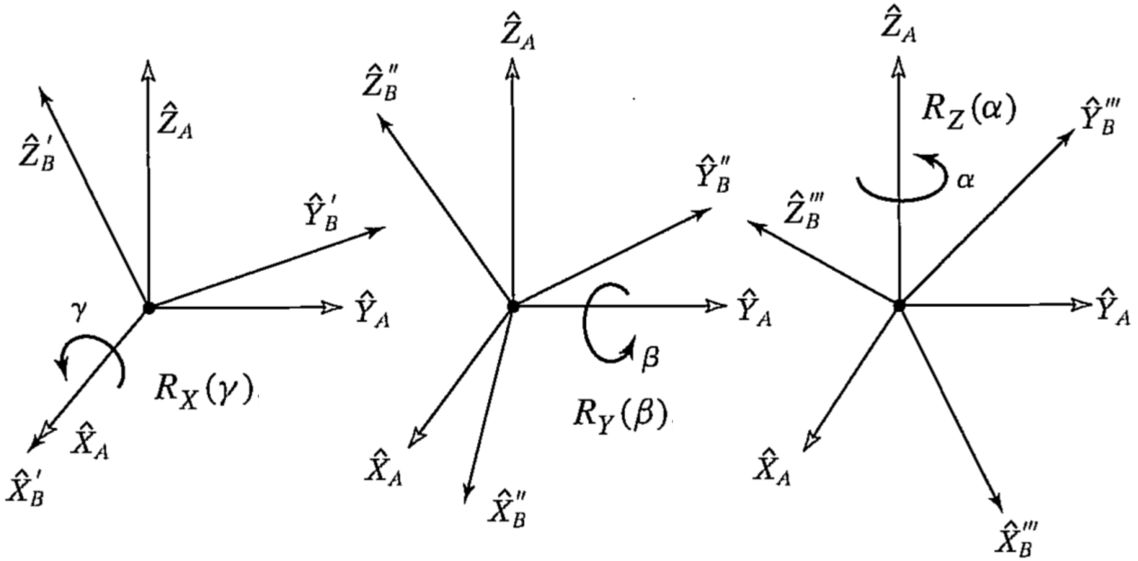
X-Y-Z Fixed Angles

Roll-Pitch-Yaw Angles



$$ {}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

$$ \beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}), $$

$$ \alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta), $$

$$ \gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta), $$

Atan2(y, x) is a two-argument arc tangent function

- Using the positive square root, we can obtain single solution between $[-\pi, \pi]$
- If $\beta = \pm 90.0°$, then we can choose the following cases.

$$ {}^A_B R_{XYZ}(\gamma, \beta, \alpha) = R_Z(\alpha) R_Y(\beta) R_X(\gamma) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

$\beta = 90.0°$, $\alpha = 0.0$, $\gamma = \text{Atan2}(r_{12}, r_{22})$.

or

$\beta = -90.0°$, $\alpha = 0.0$, $\gamma = -\text{Atan2}(r_{12}, r_{22})$

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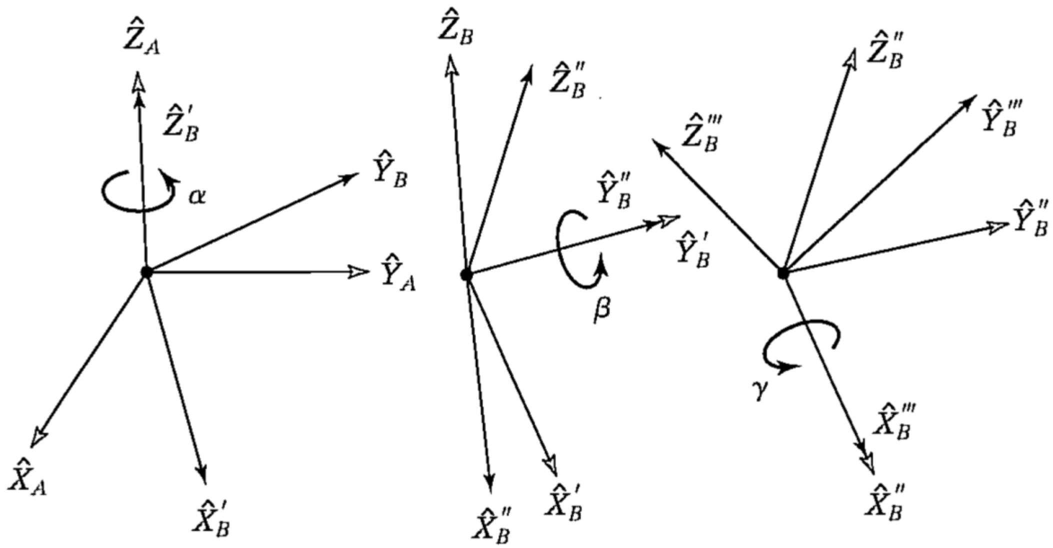
Z-Y-X Euler Angles

W.R.T. the Moving System {B} instead of the Fixed {A}



$$ {}^{A}_{B}R = {}^{A}_{B'}R \; {}^{B'}_{B''}R \; {}^{B''}_{B}R $$

Three rotations taken about fixed axes yield the same final orientation as the same three rotations taken in opposite order about the axes of the moving frame.

$$ {}^{A}_{B}R_{Z'Y'X'} = R_Z(\alpha)R_Y(\beta)R_X(\gamma) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ {}^{A}_{B}R_{Z'Y'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

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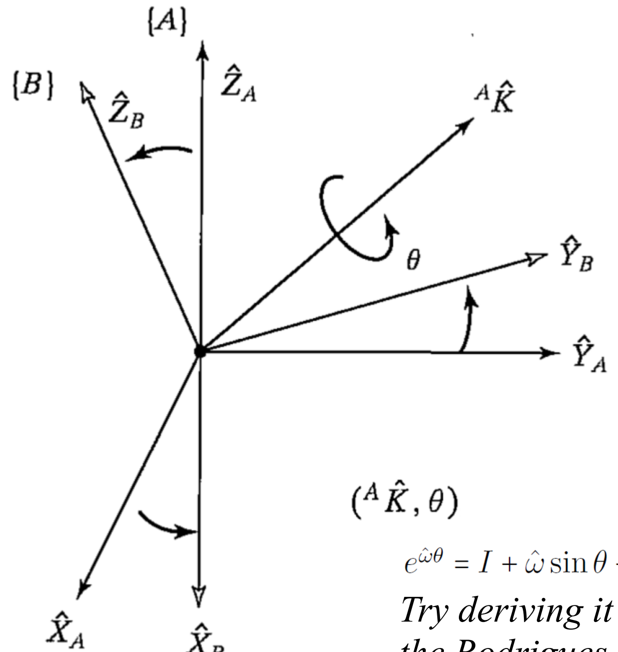
Equivalent Angle-Axis Representation

If the axis is a general direction (rather than one of the unit directions), any orientation may be obtained through proper axis and angle selection

Start with the frame coincident with a known frame $\{A\}$; then rotate $\{B\}$ about the vector ${}^A\hat{K}$ by an angle θ according to the right-hand rule.



$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

Try deriving it yourself using the Rodrigues equation

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

where $c\theta = \cos\theta$, $s\theta = \sin\theta$, $v\theta = 1 - \cos\theta$, and ${}^A\hat{K} = [k_x k_y k_z]^T$.

$${}^A_B R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\theta = \mathrm{Acos}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

- Simple form of representation
- θ falls between 0 and π
- Two solutions problem
- Becomes ill-defined for small angular rotations

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Euler Parameters as a Unit Quaternion

Evolving from Equivalent Angle-Axis to A Four-Parameter System

- If equivalent axis $\hat{K} = [k_x \; k_y \; k_z]^T$ with equivalent angle θ

$$\epsilon_1 = k_x \sin\frac{\theta}{2},$$
$$\epsilon_2 = k_y \sin\frac{\theta}{2},$$
$$\epsilon_3 = k_z \sin\frac{\theta}{2},$$
$$\epsilon_4 = \cos\frac{\theta}{2}.$$

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$$

A Unit Quaternion as a 4x1 vector

Rotation matrix written in Euler Parameters

$$R_\epsilon = \begin{bmatrix} 1-2\epsilon_2^2-2\epsilon_3^2 & 2(\epsilon_1\epsilon_2-\epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3+\epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2+\epsilon_3\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_3^2 & 2(\epsilon_2\epsilon_3-\epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3-\epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3+\epsilon_1\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_2^2 \end{bmatrix}$$

$$\epsilon_1 = \frac{r_{32}-r_{23}}{4\epsilon_4},$$
$$\epsilon_2 = \frac{r_{13}-r_{31}}{4\epsilon_4},$$
$$\epsilon_3 = \frac{r_{21}-r_{12}}{4\epsilon_4},$$
$$\epsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}.$$

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Computation Considerations

Practical Reality

- Homogeneous representation requires wasteful time multiplying by zeros and ones
  - The availability of inexpensive computing power is largely responsible for the growth of the robotics industry;
  - yet, for some time to come, efficient computation will remain an important issue in the design of a manipulation system.
- Order of multiplication
  - ${}^{A}P = {}^{A}_{D}R \, {}^{D}P$ 63 multiplications and 42 additions
  - ${}^{A}P = {}^{A}_{B}R \, {}^{B}_{C}R \, {}^{C}_{D}R \, {}^{D}P$ 27 multiplications and 18 additions

$${}^{A}P = {}^{A}_{B}R \, {}^{B}_{C}R \, {}^{C}P$$

$${}^{A}P = {}^{A}_{B}R \, {}^{B}P$$

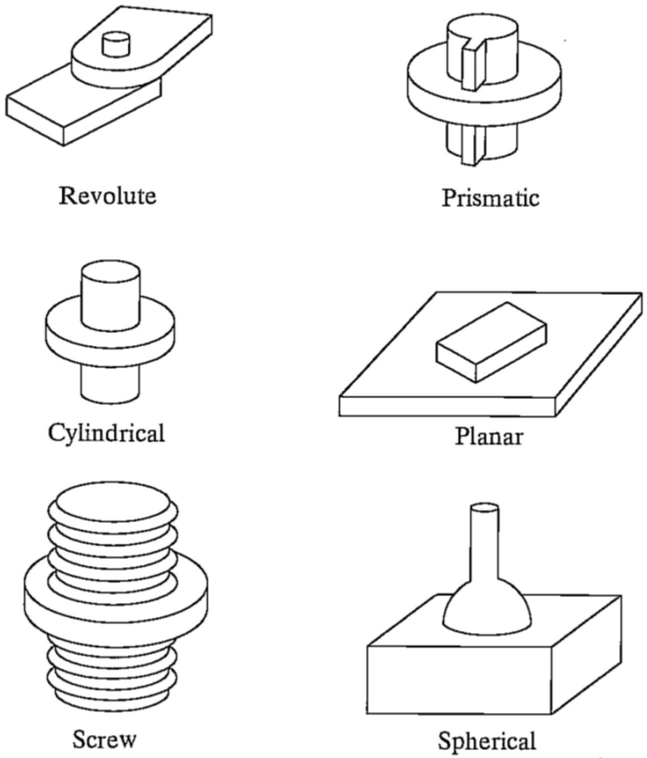
$${}^{A}P = {}^{A}P,$$

$${}^{A}P = {}^{A}_{B}R \, {}^{B}_{C}R \, {}^{C}_{D}R \, {}^{D}P.$$

Kinematics

The science of motion that treats the subject without regard to the forces that cause it

- A ***manipulator*** may be thought of as a set of bodies connected in a chain by joints
  - These bodies are called *links*.
  - *Joints* form a connection between a neighboring pair of links
    - *Lower-pair Joints*: two surfaces sliding over one another
    - Most manipulators have **revolute** or **prismatic** joints, both having 1 degree of freedom



- Manipulator Kinematics
  - All the geometrical and time-based properties of the motion
    - *The position, the velocity, the acceleration, and all higher order derivatives of the position variables (w.r.t. time or any other variable(s))*

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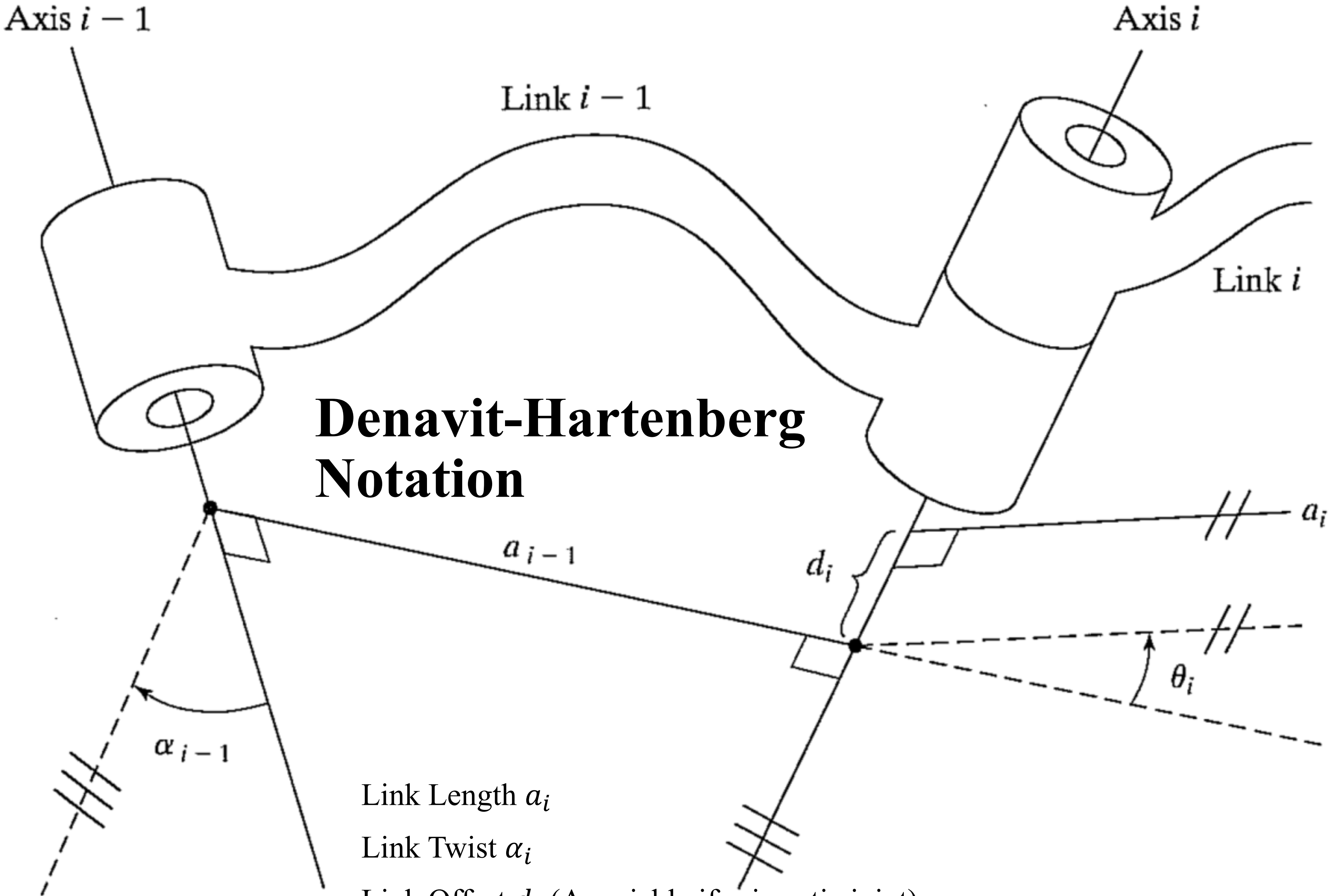

Denavit-Hartenberg Notation

Link Length a_i

Link Twist α_i

Link Offset d_i (A variable if prismatic joint)

Joint Angle θ_i (A variable if revolute joint)

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DH Parameters

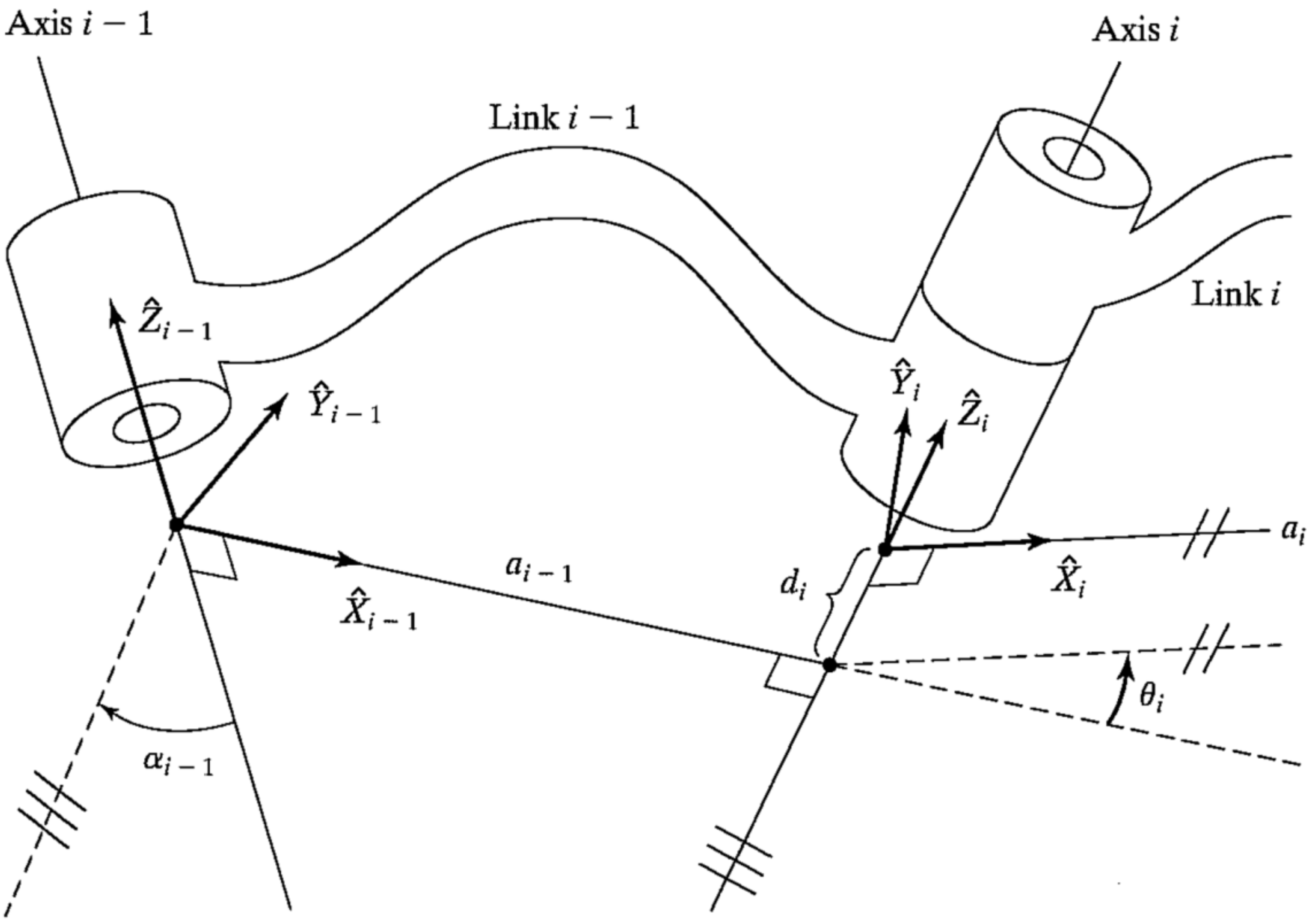
Frame Attachment

a_i = the distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$;

α_i = the angle from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured about $\hat{X}_i$;

d_i = the distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$; and

θ_i = the angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$.



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Uniqueness of Link Parameters

Practical Considerations

- Two choices of joint axis direction, $\hat{Z}_i$
- Two choices of link length axis direction, $\hat{X}_i$, for intersecting joints (i.e. $a_i = 0$)
- Arbitrary choice of origin when axes are parallel
- Freedom in frame assignment with prismatic joints

- Meaning that there could be multiple ways of writing the DH parameters, depending on the different choice of frame assignment.
  - Also multiple ways of interpreting the calculated results, if using different ways of frame assignment

- *Careful, or you might get lost*

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Procedure of Link-Frame Attachment

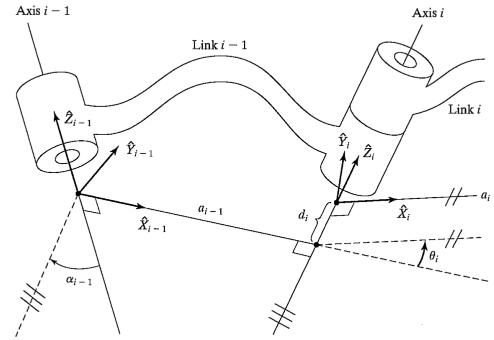
1. Identify the joint axes and imagine (or draw) infinite lines along them. For steps 2 through 5 below, consider two of these neighboring lines (at axes i and $i+1$).
2. Identify the common perpendicular between them, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the ith axis, assign the link-frame origin.
3. Assign the $\hat{Z}_i$ axis pointing along the ith joint axis.
4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes.
5. Assign the $\hat{Y}_i$ axis to complete a right-hand coordinate system.
6. Assign {0} to match {1} when the first joint variable is zero. For $\{N\}$, choose an origin location and $\hat{X}_N$ direction freely, but generally so as to cause as many linkage parameters as possible to become zero.

$a_i =$ *the distance from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured along* $\hat{X}_i$;

$\alpha_i =$ *the angle from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured about* $\hat{X}_i$;

$d_i =$ *the distance from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured along* $\hat{Z}_i$; and

$\theta_i =$ *the angle from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured about* $\hat{Z}_i$.



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Exercise

O => Z => X => a => α => d => θ

1. Identify the joint axes and imagine (or draw) infinite lines along them. For steps 2 through 5 below, consider two of these neighboring lines (at axes i and $i+1$).
2. Identify the common perpendicular between them, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the ith axis, assign the link-frame origin.
3. Assign the $\hat{Z}_i$ axis pointing along the ith joint axis.
4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes.
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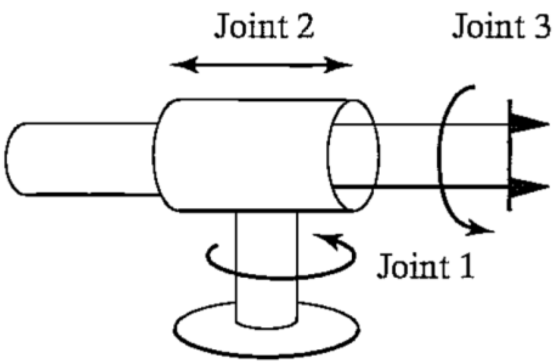

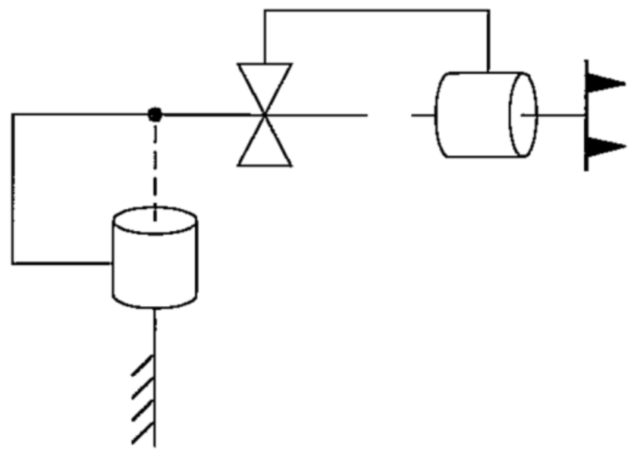

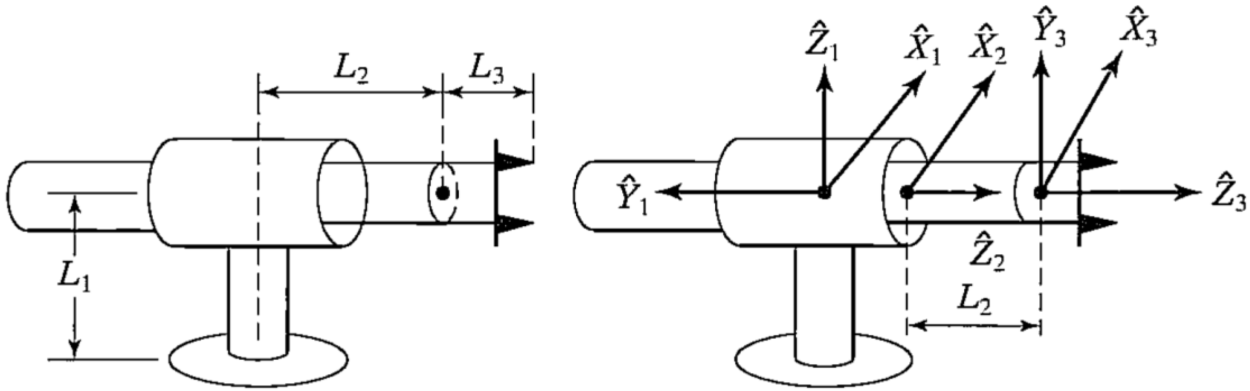

(a) (b)

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90°	0	d_2	0
3	0	0	L_2	θ_3

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Derivation of Link Transformations

Each of the four transformations will be a function of one link parameter only and will be simple enough that we can write down its form by inspection

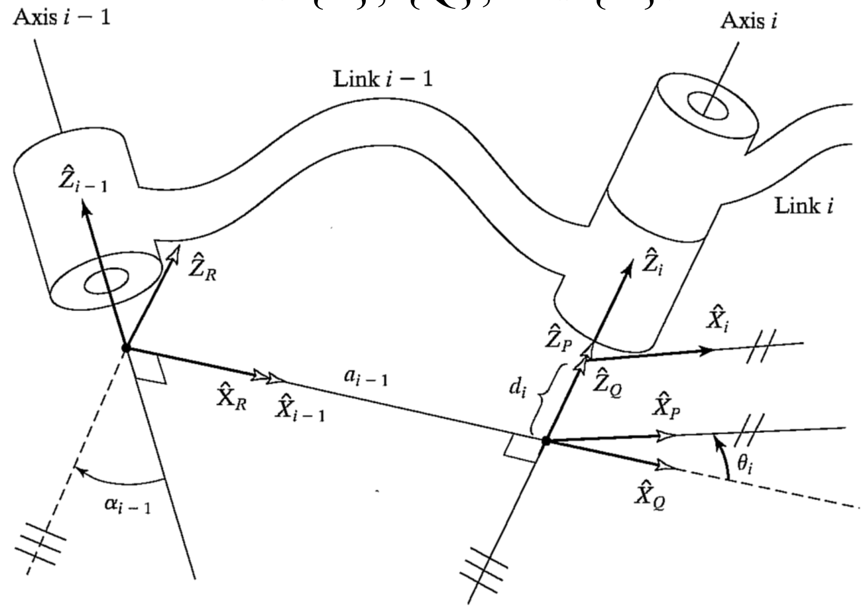
$$ {}^{i-1}_{i}T = {}^{i-1}_{R}T \, {}^{R}_{Q}T \, {}^{Q}_{P}T \, {}^{P}_{i}T $$

$$ {}^{i-1}_{i}T = R_X(\alpha_{i-1}) D_X(a_{i-1}) R_Z(\theta_i) D_Z(d_i) $$

$$ {}^{i-1}_{i}T = \mathrm{Screw}_X(a_{i-1}, \alpha_{i-1}) \, \mathrm{Screw}_Z(d_i, \theta_i) $$

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Location of intermediate frames {P}, {Q}, and {R}.



Concatenating link transformations

- The single transformation that relates frame {N} to frame {0}

$$ {}^{0}_{N}T = {}^{0}_{1}T \, {}^{1}_{2}T \, {}^{2}_{3}T \ldots {}^{N-1}_{N}T $$

a function of all n joint variables

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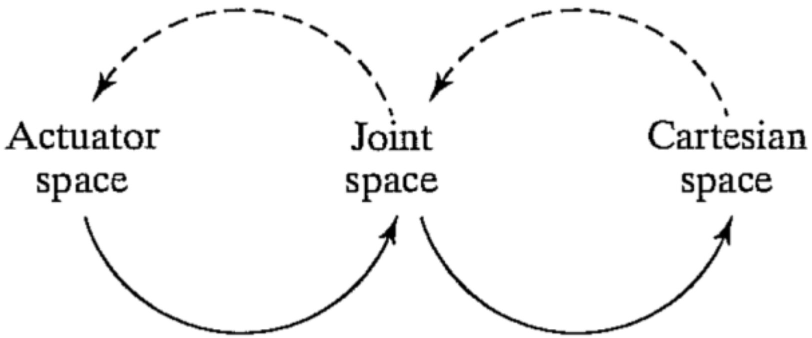
Actuator Space, Joint Space, and Cartesian Space



- **Joint Space**
  - The space of all joint vectors.
  - A n×1 joint vector refers to a set of n joint variables specifying the position of all the links of a manipulator of n degrees of freedom.
- **Cartesian Space**
  - When position is measured along orthogonal axes and orientation is measured according to any Cartesian conventions.
- **Actuator Space**
  - The space of all actuator positions.
  - Computations necessary to realized the joint vector as a function of a set of actuator values.

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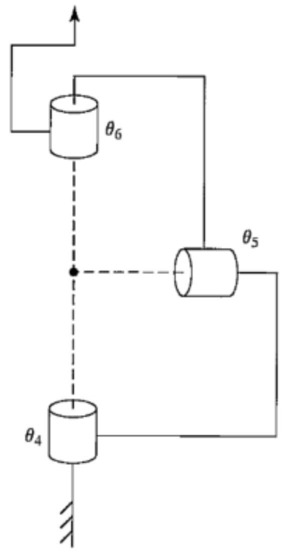
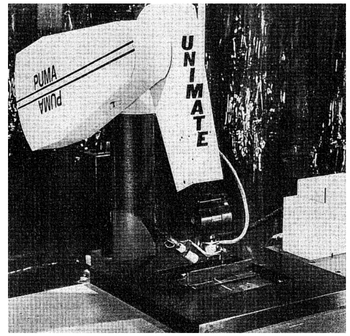
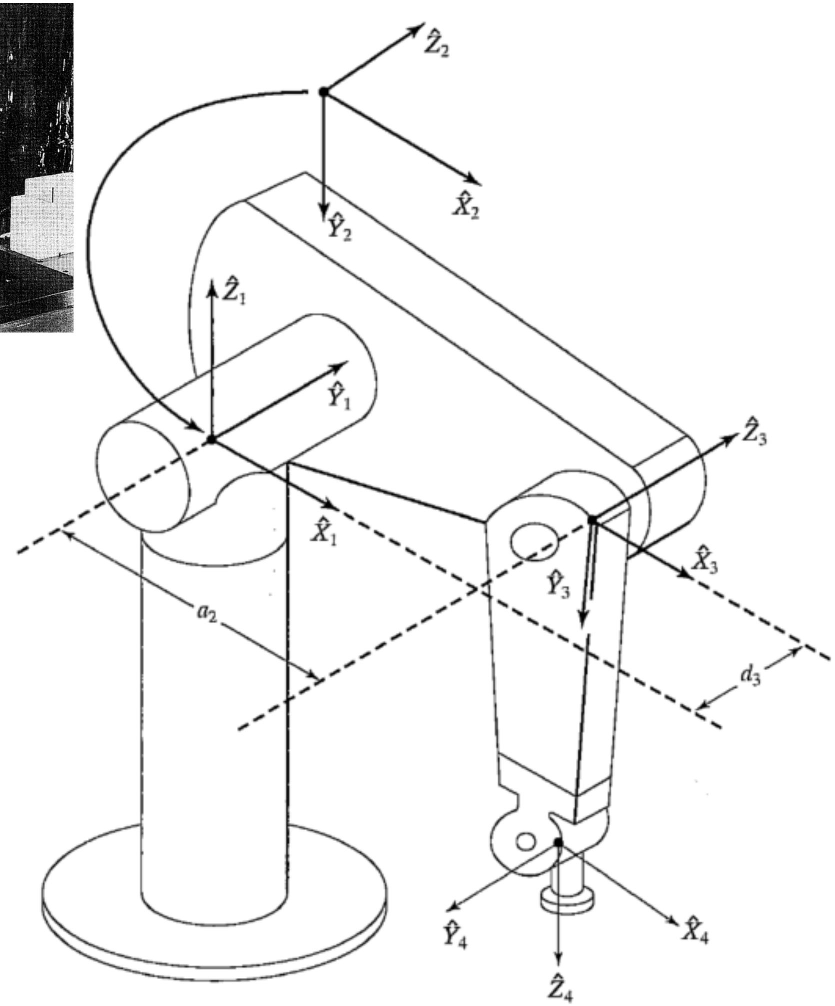
Example: Kinematics of PUMA 560

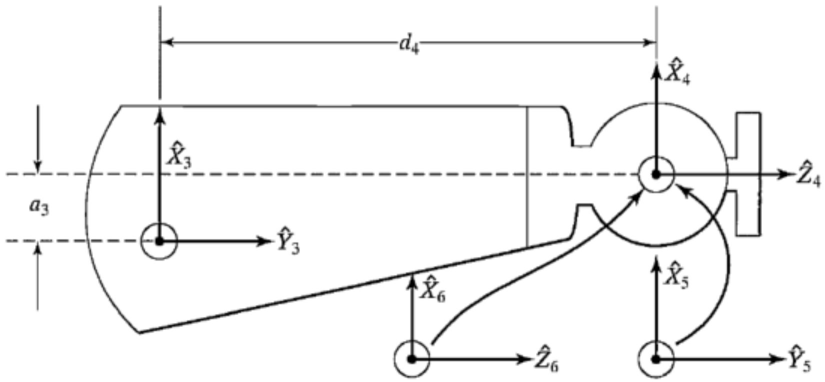
Attach the Frames => Determine the DH => Check & Revise

i	$\alpha_i - 1$	$a_i - 1$	d_i	θi
1	0	0	0	θ_1
2	−90°	0	0	θ_2
3	0	a_2	d_3	θ_3
4	−90°	a_3	d_4	θ_4
5	90°	0	0	θ_5
6	−90°	0	0	θ_6









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$$
{}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^2_3T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^3_4T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^4_5T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^5_6T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

$$
{}^4_6T = {}^4_5T\,{}^5_6T = \begin{bmatrix} c_5c_6 & -c_5s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5c_6 & -s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^3_6T = {}^3_4T\,{}^4_6T = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & a_3 \\ s_5c_6 & -s_5s_6 & c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^1_3T = {}^1_2T\,{}^2_3T = \begin{bmatrix} c_{23} & -s_{23} & 0 & a_2c_2 \\ 0 & 0 & 1 & d_3 \\ -s_{23} & -c_{23} & 0 & -a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^1_6T = {}^1_3T\,{}^3_6T = \begin{bmatrix} {}^1r_{11} & {}^1r_{12} & {}^1r_{13} & {}^1p_x \\ {}^1r_{21} & {}^1r_{22} & {}^1r_{23} & {}^1p_y \\ {}^1r_{31} & {}^1r_{32} & {}^1r_{33} & {}^1p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
{}^1r_{11} &= c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6, \\
{}^1r_{21} &= -s_4c_5c_6 - c_4s_6, \\
{}^1r_{31} &= -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6, \\
{}^1r_{12} &= -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6, \\
{}^1r_{22} &= s_4c_5s_6 - c_4c_6, \\
{}^1r_{32} &= s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6, \\
{}^1r_{13} &= -c_{23}c_4s_5 - s_{23}c_5, \\
{}^1r_{23} &= s_4s_5, \\
{}^1r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\
{}^1p_x &= a_2c_2 + a_3c_{23} - d_4s_{23}, \\
{}^1p_y &= d_3, \\
{}^1p_x &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
$$

$$
{}^0_6T = {}^0_1T\,{}^1_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
r_{11} &= c_1[c_{23}(c_4c_5c_6 - s_4s_5) - s_{23}s_5c_5] + s_1(s_4c_5c_6 + c_4s_6), \\
r_{21} &= s_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6 - c_1(s_4c_5c_6 + c_4s_6), \\
r_{31} &= -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6,
\end{aligned}
$$

$$
\begin{aligned}
r_{12} &= c_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] + s_1(c_4c_6 - s_4c_5s_6), \\
r_{22} &= s_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5s_6), \\
r_{32} &= -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6,
\end{aligned}
$$

$$
\begin{aligned}
r_{13} &= -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5, \\
r_{23} &= -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5, \\
r_{33} &= s_{23}c_4s_5 - c_{23}c_5,
\end{aligned}
$$

$$
\begin{aligned}
p_x &= c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1, \\
p_y &= s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_3c_1, \\
p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
$$

i	$\alpha_i - 1$	$a_i - 1$	d_i	θi
1	0	0	0	θ_1
2	−90°	0	0	θ_2
3	0	a_2	d_3	θ_3
4	−90°	a_3	d_4	θ_4
5	90°	0	0	θ_5
6	−90°	0	0	θ_6

The basic equations for all kinematic analysis of this manipulator

Attach the frames wisely to ease the computational burden

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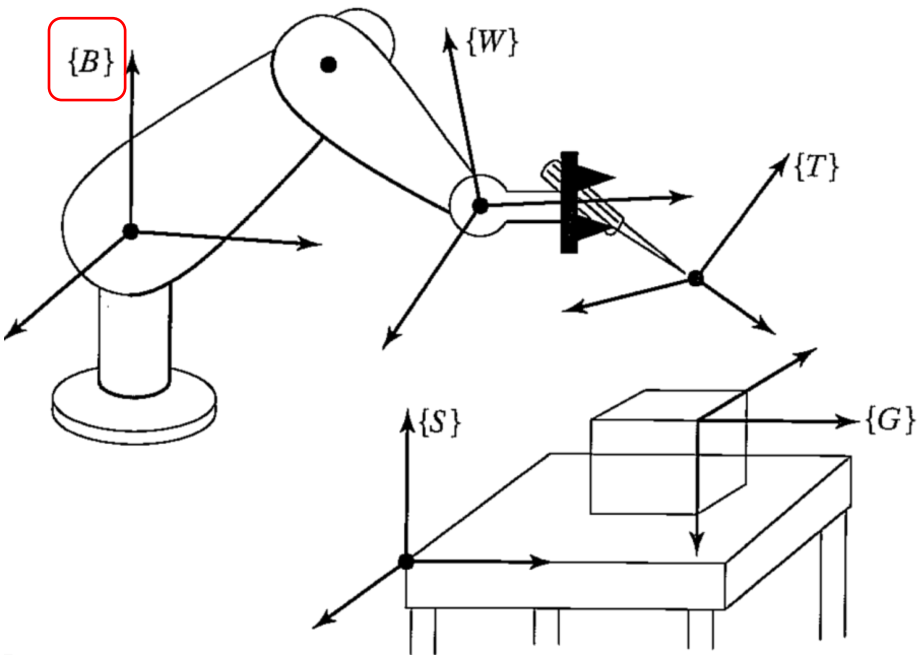
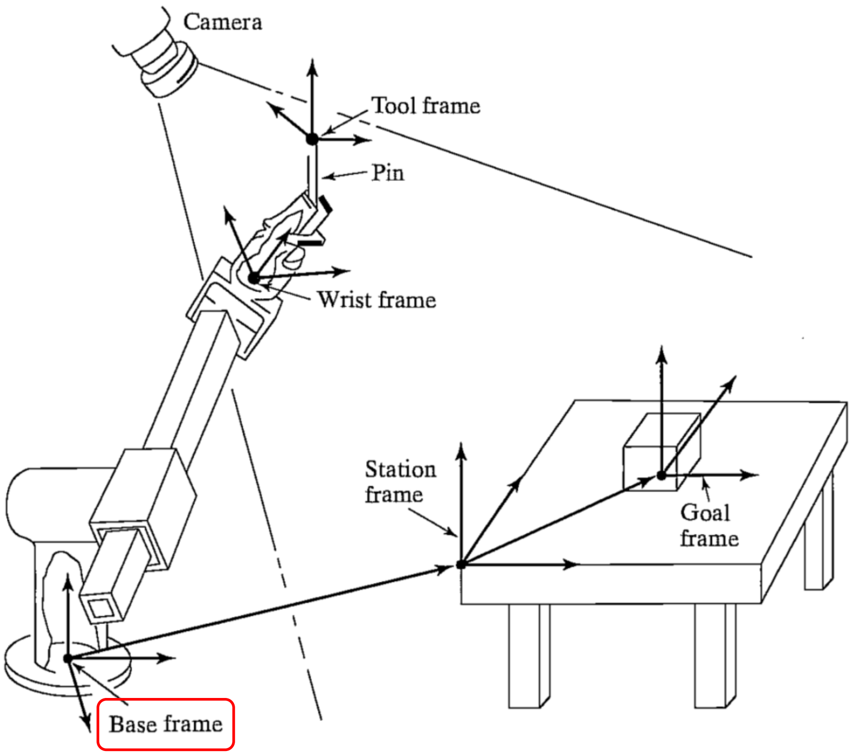
The Base Frame, {B}

Frames with Standard Names

- {B} is located at the base of the manipulator.
  - It is merely another name for frame {0}.
- It is affixed to a nonmoving part of the robot, sometimes called link 0.

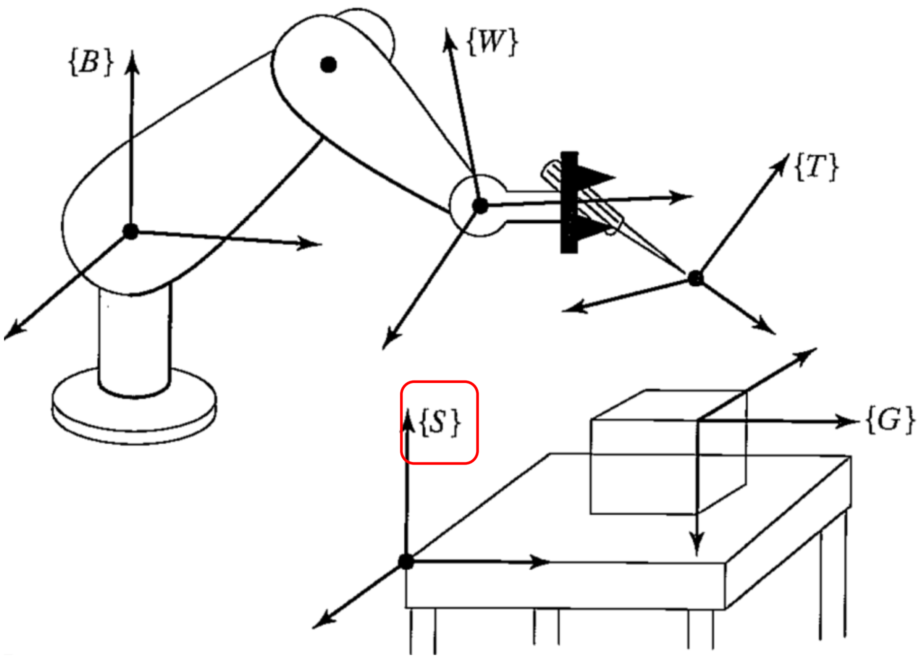


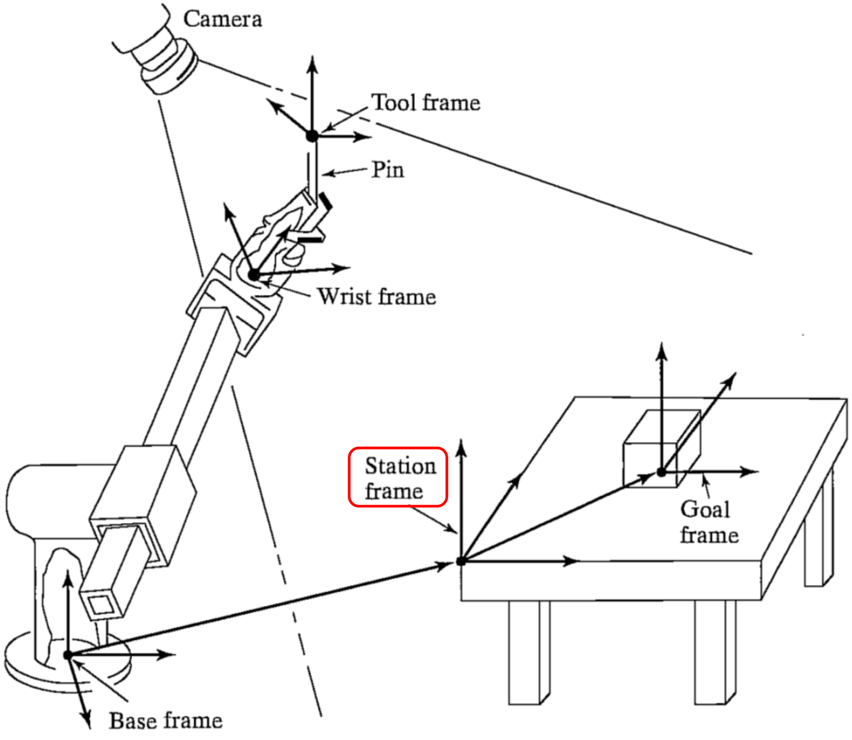


The Station Frame, {S}

Frames with Standard Names

- {S} is located in a task-relevant location. (Task/Work Frame/Universe Frame)
  - In the following figure, it is at the corner of a table upon which the robot is to work.
- As far as the user of this robot system is concerned, {S} is the universe frame, and all actions of the robot are performed relative to it.

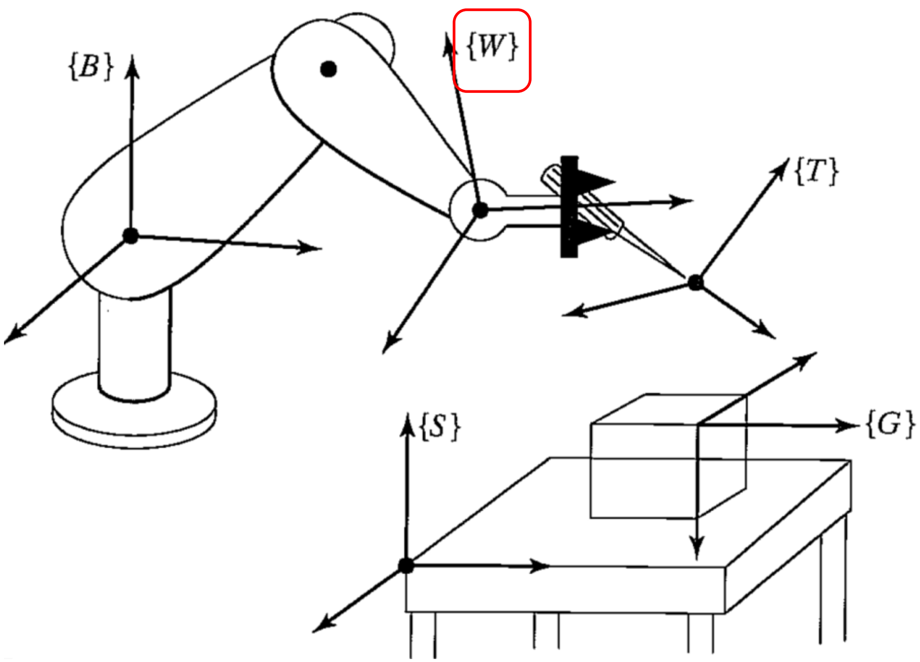


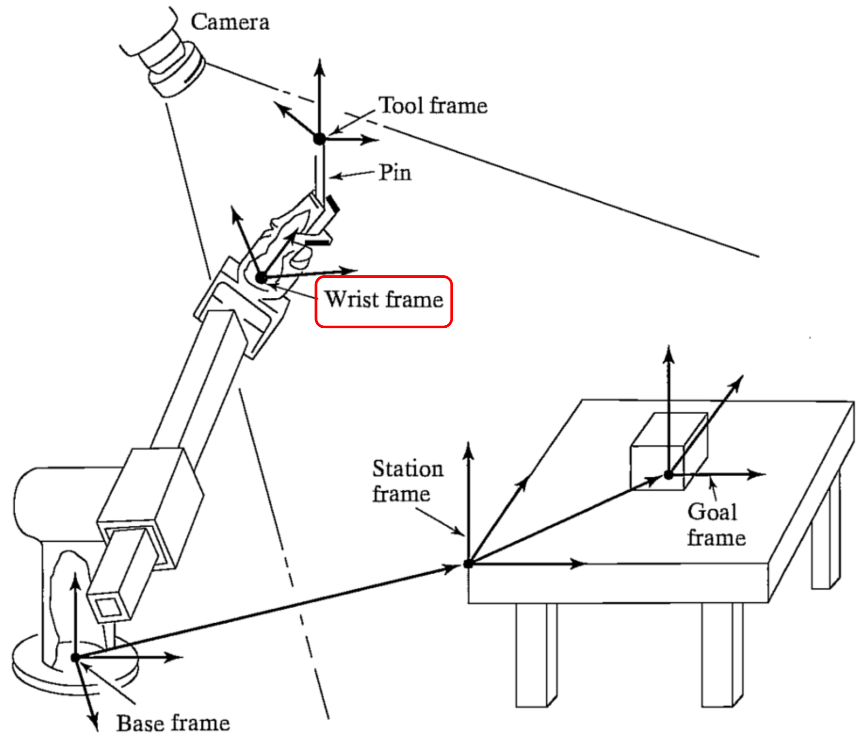


The Wrist Frame, {W}

Frames with Standard Names

- {W} is affixed to the last link of the manipulator.
  - It is another name for frame {N}, the link frame attached to the last link of the robot.
- Very often, {W} has its origin fixed at a point called the wrist of the manipulator, and {W} moves with the last link of the manipulator. It is defined relative to the base frame.

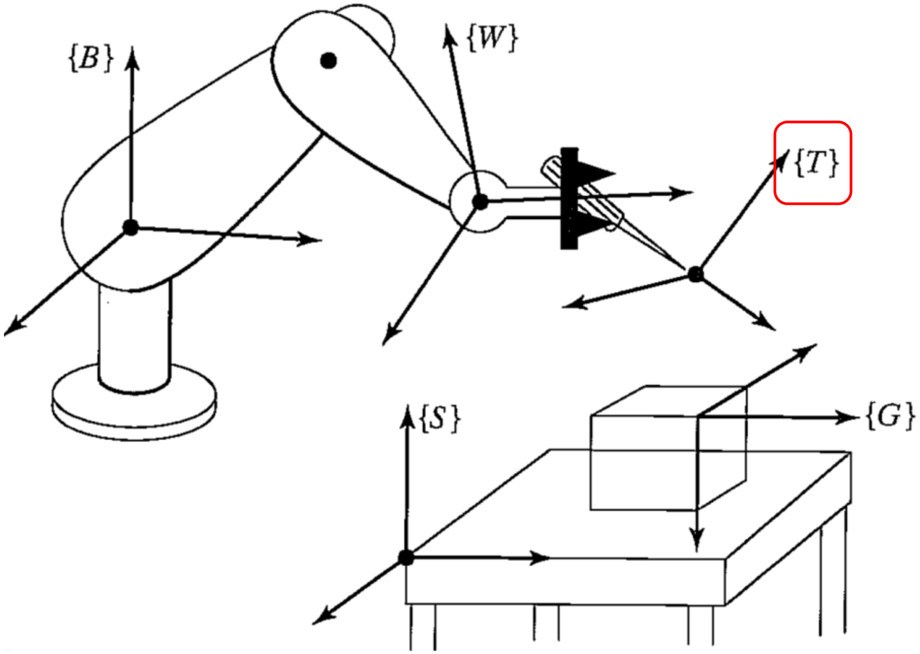


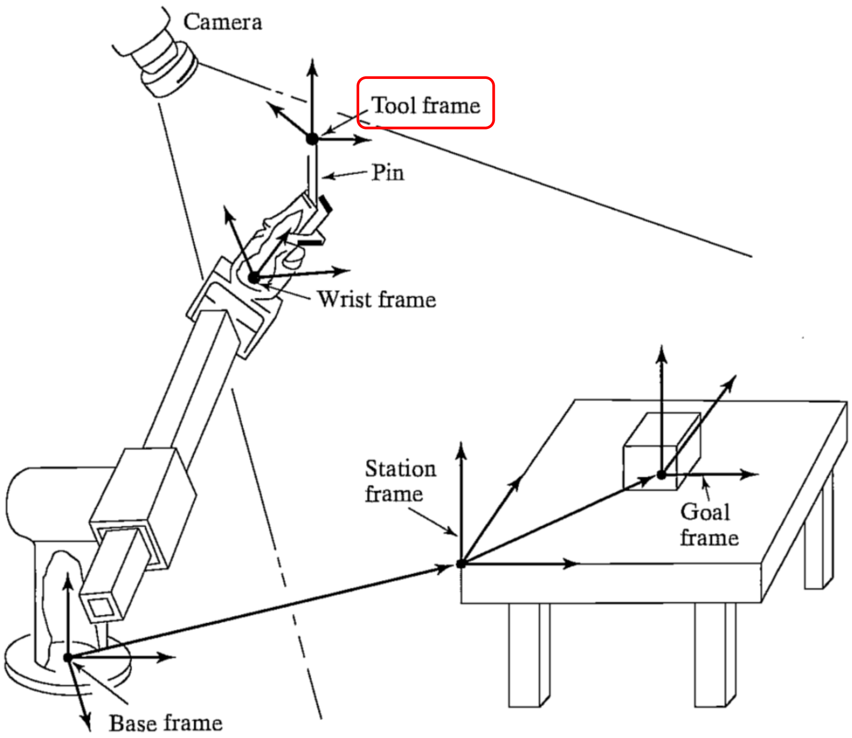


The Tool Frame, {T}

Frames with Standard Names

- {T} is affixed to the end of any tool the robot happens to be holding.
  - When the hand is empty, {T} is usually located with its origin between the robot fingertips.
- The tool frame is always specified with respect to the wrist frame.
  - In the following example, the tool frame is defined with its origin at the tip of a pin that the robot is holding.

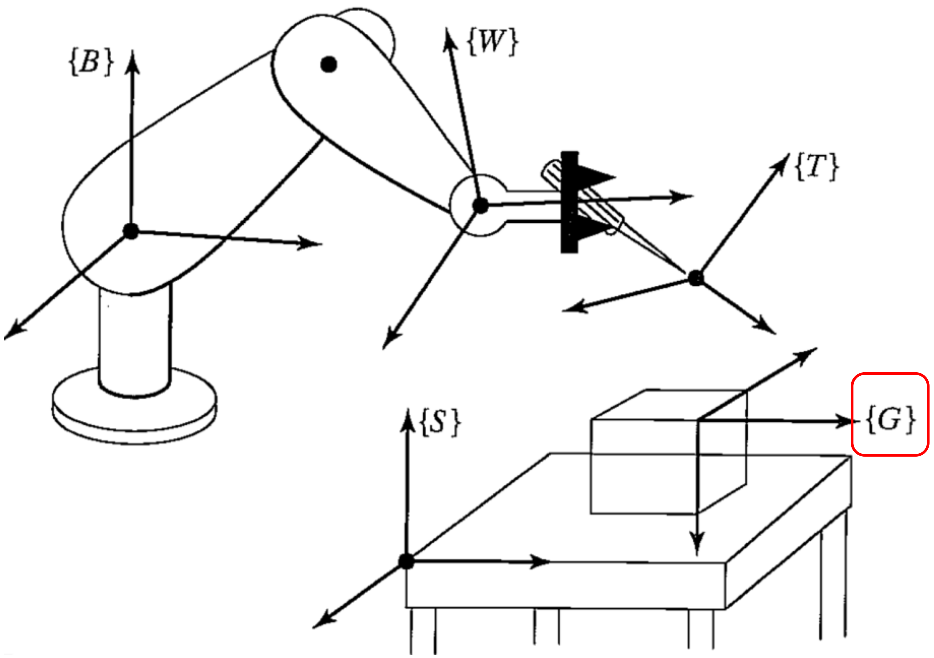


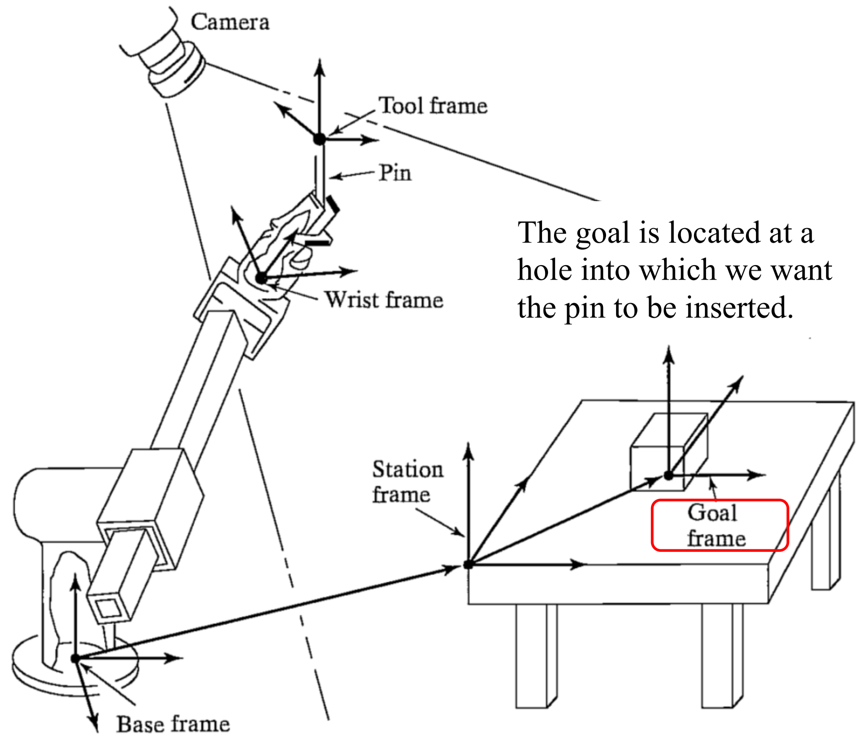


The Goal Frame, {G}

Frames with Standard Names

- {G} is a description of the location to which the robot is to move the tool.
  - Specifically this means that, at the end of the motion, the tool frame should be brought to coincidence with the goal frame.
- {G} is always specified relative to the station frame.





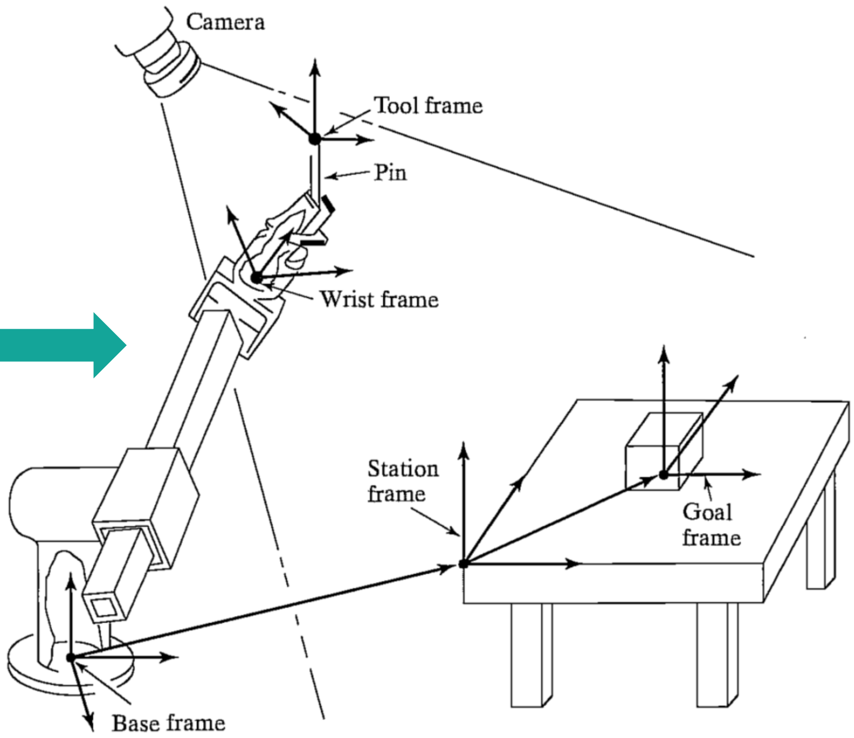
Where is the Tool?

We wish to calculate the value of the tool frame, {T}, relative to the station frame, {S}.

- The position and orientation of the tool it is holding (or of its empty hand) with respect to a convenient coordinate system.

$$ {}^{S}_{T}T = {}^{B}_{S}T^{-1}\, {}^{B}_{W}T\, {}^{W}_{T}T. $$

- **WHERE** function in some robot systems
  - It computes "where" the arm is.
  - The position and orientation of the pin relative to the table top



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Computation Considerations

In many practical manipulator systems, the time required to perform kinematic calculations is a consideration

- **The use of fixed- or floating-point representation of the quantities involved.**
  - Fixed over floating as the limited dynamic range of the variables
  - Roughly estimated a 24 fixed-point representation is enough.
- **Avoid computing common terms over and over throughout the computation**
  - Factoring equations of the transformation matrix to reduce the number of multiplications and additions at the cost of creating local variables (usually a good trade-off)
- **The calculation of the transcendental functions (sine and cosine) is a major expense in kinematics calculations**
  - Table-lookup implementations of the transcendental functions instead of actual calculations
- **Redundant computation of the kinematics as nine quantities are calculated to represent orientation**
  - One way is to calculate only two columns of the rotation matric and then to compute a cross product (requiring only six multiplications and three additions) to compute the third column.
  - Choose the two least complicated columns to compute.

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Week 05 | Lecture 05
Mathematical Foundations

Thank you~

Wan Fang
Southern University of Science and Technology